

On Kummer's test of convergence and its relation to basic comparison tests

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Abstract

Testing convergence of a series $\sum a_n$ is an important part of scientific areas. A very basic comparison test bounds the terms of $\sum a_n$ with the terms of some known convergent series $\sum b_n$ (either in the form $a_n \leq b_n$ or $a_{n+1}/a_n \leq b_{n+1}/b_n$). In 19th century Kummer proposed a test of convergence for any positive series saying that the series $\sum a_n$ converges if and only if there is a positive series $\sum p_n$ and a real constant $c > 0$ such that $p_n(a_n/a_{n+1} - p_{n+1}) \geq c$. Furthermore, by choosing specific parameters p_n , one can obtain other tests like Raabe's, Gauss' or Bertrand's as special cases. In 1995 Samelson noted that there is another interesting relation between Kummer's test and basic comparison tests, particularly, that one can easily transform the numbers p_n to numbers b_n , and he sketched a simple proof of this statement. In this paper we give a full formal proof of this statement using a (slightly) different approach.

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1 Introduction

In the theory of infinite series the Kummer's test of convergence/divergence has an important place because it can determine the character of any positive series. Additionally, tests like Raabe's, Gauss' or Bertrand's can be viewed as its special cases obtained by choosing specific parameters p_n 's. The test was first given by German mathematician Ernst Kummer in 1835 [3] and later improved by Ulisse Dini [1]. More recently, a short and simple proof of this test was given by Tong [5].

The comparison of Kummer's test with the basic comparison tests (introduced explicitly by Cauchy in 1821, see *Section 2* for more details) first appeared in Samelson [4]. Samelson in his proof uses the fact that all positive series $\sum b_n$ can be written in the form $\sum (c_{n-1} - c_n)$, where $c_n = \sum_{i=1}^n b_i$. In our paper we give the proof of the same statement, namely that *Kummer's test is equal to the basic comparison test*, by using a different approach. Moreover, as Samelson only sketched his proof, we provide a full formal proof.

For the sake of completeness, we state a few basic theorems from the theory of infinite series (*Section 2*) as well as proofs of two lemmas used in the main proof (*Section 3*). Note that, unless stated otherwise, the expression for a positive series $\sum_{n=1}^{\infty} a_n$ will be abbreviated as $\sum a_n$.

2 Elementary theorems

In this section we provide the most basic theorems in convergence of positive series which will be referenced in the next section.

Theorem 2.1 (Cauchy-Bolzano). *The series $\sum a_n$ converges if and only if*

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \in \mathbb{N} \ n > N, \forall p \in \mathbb{N} : |a_n + a_{n+1} + \dots + a_{n+p}| < \epsilon.$$

Corollary 2.2. *If series $\sum a_n$ converges, then $\lim_{n \rightarrow \infty} a_n = 0$.*

Corollary 2.3. *If series $\sum a_n$ converges, then $\forall \epsilon > 0, \exists k \in \mathbb{N} : \left| \sum_{n=k+1}^{\infty} a_n \right| < \epsilon$.*

Theorem 2.4 (Basic comparison test 1). *Let $\sum a_n$ and $\sum b_n$ be two series with positive terms. Let at most finite count of numbers $n \in \mathbb{N}$ fail the inequality $a_n \leq b_n$. Then, the convergence of series $\sum b_n$ implies the convergence of series $\sum a_n$, and the divergence of series $\sum a_n$ implies the divergence of series $\sum b_n$.*

Theorem 2.5 (Basic comparison test 2). *Let $\sum a_n$ and $\sum b_n$ be two series with positive terms. Let at most finite count of numbers $n \in \mathbb{N}$ fail the inequality $\frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n}$. Then, the convergence of series $\sum b_n$ implies the convergence of series $\sum a_n$, and the divergence of series $\sum a_n$ implies the divergence of series $\sum b_n$.*

3 Kummer's equivalence with basic comparison tests

To demonstrate the equivalence of Kummer's test with the basic convergence tests from *Section 2* we first state two necessary lemmas (adopted from the exercises from [2]).

Lemma 3.1. *If $\sum a_n$ is a convergent series with positive terms, then there exists a monotonous sequence $\{B_n\}_{n=1}^{\infty}$ such that $\lim_{n \rightarrow \infty} B_n = \infty$ and $\sum a_n B_n$ converges.*

Proof. Let $\sum a_n$ be a convergent series with positive terms. From *Corollary 2.2* it follows that $\lim_{n \rightarrow \infty} \frac{1}{a_n} = \infty$, and from *Corollary 2.3* we construct an increasing subsequence of natural numbers $\{\xi_n\}_{n=1}^{\infty}$ such that

$$\left(\sum_{k > \xi_n} a_k \right) \frac{1}{a_n} < a_n. \quad (3.1)$$

We define the numbers B_n as follows:

$$B_n = \begin{cases} 0 & : \text{if } n \in [1, \xi_1] \\ \frac{1}{a_k} & : \text{if } n \in [\xi_k, \xi_{k+1}). \end{cases}$$

The convergence of $\sum a_n B_n$ follows from (3.1):

$$\sum_{n=1}^{\infty} a_n B_n = \sum_{n=1}^{\xi_1-1} a_n 0 + \sum_{n=1}^{\infty} \left(\frac{1}{a_n} \sum_{k=\xi_n}^{\xi_{n+1}-1} a_k \right) \leq \sum_{n=1}^{\infty} a_n.$$

□

Lemma 3.2. *If $\sum a_n$ is a divergent series with positive terms, then there exists a monotonous sequence $\{B_n\}_{n=1}^{\infty}$ such that $\lim_{n \rightarrow \infty} B_n = 0$ and $\sum a_n B_n$ diverges.*

Proof. Let $\sum a_n$ be a divergent series with positive terms. We will consider two cases. First, if $\limsup_{n \rightarrow \infty} a_n \geq \epsilon > 0$, then by leaving out all terms $a_n < \frac{\epsilon}{2}$ we will not change the divergent character of the series. Let $\{\xi_k\}_{n=1}^{\infty}$ denote all indices n such that $a_n \geq \frac{\epsilon}{2}$ (in the increasing order). We define the numbers B_n as follows:

$$B_n = \begin{cases} 0 & : n \neq \xi_k \text{ for all } k \\ \frac{1}{k} & : n = \xi_k \text{ for some } k. \end{cases}$$

Obviously, $\lim_{n \rightarrow \infty} B_n = 0$, and

$$\sum_{n=1}^{\infty} a_n B_n \geq \sum_{n=1}^{\infty} \frac{\epsilon}{2n} = \infty.$$

Second, if $\lim_{n \rightarrow \infty} a_n = 0$, then from the divergence of the series $\sum a_n$ we have an increasing subsequence $\{\xi_n\}_{n=1}^{\infty}$ of natural numbers such that

$$\sum_{k=\xi_n}^{\xi_{n+1}-1} a_k > n, \quad \xi_1 = 1. \quad (3.2)$$

We define the numbers B_n as follows:

$$B_n = \frac{1}{k} \text{ for } n \in [\xi_k, \xi_{k+1}) \text{ and } k \in \mathbb{N}$$

By (3.2) the series $\sum a_n B_n$ diverges because

$$\sum_{n=1}^{\infty} a_n B_n = \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=\xi_n}^{\xi_{n+1}-1} a_k \right) \geq \sum_{n=1}^{\infty} 1 = \infty.$$

□

Now we state and prove Kummer's test and thus demonstrate its equality with comparison tests. The statement of the theorem is adopted from Tong [5].

Theorem 3.3 (Kummer's test [5]). *Let $\sum a_n$ be a positive series.*

(1) *$\sum a_n$ is convergent if and only if there is a positive series $\sum p_n$ and a real number $c > 0$, such that $p_n(a_n/a_{n+1}) - p_{n+1} \geq c$.*

(2) *$\sum a_n$ is divergent if and only if there is a positive series $\sum p_n$ and a real number $c > 0$, such that $\sum 1/p_n$ diverges and $p_n(a_n/a_{n+1}) - p_{n+1} \leq 0$.*

Proof. Sufficiency.

(1) Suppose that $\sum p_n$ is a positive series and there is a real number $c > 0$, such that $p_n(a_n/a_{n+1}) - p_{n+1} \geq c$. The inequalities

$$p_n a_n - p_{n+1} a_{n+1} \geq c a_{n+1} > 0$$

implies that the sequence $\{p_n a_n\}_{n=1}^{\infty}$ is positive and decreasing; therefore, it has a limit. Furthermore, we can construct sequence of numbers $\{B_n\}_{n=1}^{\infty}$, $B_n \geq 1$ for all n , such that

$$p_n a_n - p_{n+1} a_{n+1} = c B_{n+1} a_{n+1}.$$

Thus, the series

$$\sum_{n=1}^{\infty} a_{n+1} B_{n+1} = \frac{1}{c} \sum_{n=1}^{\infty} (p_n a_n - p_{n+1} a_{n+1}) = \frac{p_1 a_1}{c} - \frac{1}{c} \lim_{n \rightarrow \infty} p_n a_n > 0$$

converges, and because $B_n \geq 1$, so does the series $\sum a_n$ according to the *First comparison test* (Theorem 2.4).

(2) Suppose that $\sum p_n$ is a positive series for which $\sum 1/p_n$ diverges and $p_n(a_n/a_{n+1}) - p_{n+1} < 0$. This inequality immediately yields

$$\frac{1/p_{n+1}}{1/p_n} \leq \frac{a_{n+1}}{a_n},$$

and since $\sum 1/p_n$ diverges, so does $\sum a_n$ according to *Second comparison test* (Theorem 2.5).

Necessity.

(1) Suppose that $\sum a_n$ is a convergent positive series. From *Lemma 3.1* we have a positive monotonous sequence $\{B_n\}_{n=1}^\infty$, such that $\lim_{n \rightarrow \infty} B_n = \infty$ and $\sum a_n B_n$ converges. Let p_1 be a positive number such that $\sum a_n B_n = p_1 a_1$. We define the sequence $\{p_n\}_{n=1}^\infty$ recursively as follows:

$$p_{n+1}a_{n+1} = p_n a_n - a_{n+1}B_{n+1}.$$

The numbers p_n are positive because

$$\lim_{n \rightarrow \infty} p_n a_n = p_1 a_1 - \lim_{n \rightarrow \infty} \sum_{k=1}^n a_{k+1} B_{k+1} = 0$$

and the numbers $\{a_n\}$ are positive as well. Furthermore, for any $c > 0$ and sufficiently large n we have $p_n a_n - p_{n+1} a_{n+1} = a_{n+1} B_{n+1} \geq a_{n+1} c$.

(2) Suppose that $\sum a_n$ is a divergent positive series. From *Lemma 3.2* we have a positive monotonous sequence $\{B_n\}_{n=1}^\infty$, such that $\lim_{n \rightarrow \infty} B_n = 0$ and $\sum a_n B_n$ diverges. Let $p_n = 1/a_n B_n$. It follows that

$$\frac{a_{n+1} B_{n+1}}{a_n B_n} = \frac{\frac{1}{p_{n+1}}}{\frac{1}{p_n}} \leq \frac{a_{n+1}}{a_n}$$

which, according to the *Second comparison test (Theorem 2.5)*, implies the divergence of the positive series $\sum \frac{1}{p_n}$, as well as the inequality $p_n a_n - p_{n+1} a_{n+1} \leq 0$. \square

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